# Density nonlinearities in field theories for a toy model of fluctuating nonlinear hydrodynamics of supercooled liquids

Joonhyun Yeo[*]

*Division of Quantum Phases and Devices, School of Physics, Konkuk University, Seoul 143-701, Korea*
(Received 16 July 2009; published 3 November 2009)

We study a zero-dimensional version of the fluctuating nonlinear hydrodynamics (FNH) of supercooled liquids originally investigated by Das and Mazenko (DM) [Shankar P. Das and Gene F. Mazenko Phys. Rev. A **34**, 2265 (1986)]. The time-dependent density-like and momentum-like variables are introduced with no spatial degrees of freedom in this toy model. The structure of nonlinearities takes the similar form to the original FNH, which allows one to study in a simpler setting the issues raised recently regarding the field theoretical approaches to glass forming liquids. We study the effects of density nonlinearities on the time evolution of correlation and response functions by developing field theoretic formulations in two different ways: first by following the original prescription of DM and then by constructing a dynamical action which possesses a linear time-reversal symmetry as proposed recently. We show explicitly that, at the one-loop order of the perturbation theory, the DM-type field theory does not support a sharp ergodic-nonergodic transition, while the other admits one. The simple nature of the toy model in the DM formulation allows us to develop numerical solutions to a complete set of coupled dynamical equations for the correlation and response functions at the one-loop order.

## I. INTRODUCTION

The slow dynamics of supercooled liquids near the glass transition has been under intense theoretical and experimental investigation for many years. Among many theoretical attempts to understand the slowing down of supercooled liquids, the mode coupling theory (MCT) [1–4] stands out as one of the most successful ones. It explains, for example, an elaborate sequence of time relaxation processes with characteristic exponents which are consistent with experimental findings. In its initial form [5–7], referred to as the standard MCT, it predicts a sharp ergodic-to-nonergodic (ENE) transition at a critical temperature or density with the nonergodic phase characterized by the density autocorrelation function approaching a nonzero value called the nonergodicity parameter in the long time limit. Many experiments and numerical simulations, however, show that this feature is not realized in real supercooled liquids in finite dimensions, and that the ergodicity is restored at finite temperature.

There have been many attempts to put the MCT into a field theoretic framework [8–17], since it has many advantages including the possibility of a systematic improvement. Das and Mazenko (DM) [8] studied the nonlinear feedback mechanism of density fluctuations in supercooled liquids by formulating a field theoretic renormalized perturbation theory of the fluctuating nonlinear hydrodynamics (FNH) of compressible fluids. They find that the sharp transition is cutoff and the system remains ergodic at all temperatures or densities. Recently, however, the validity of the DM results, especially of those on the explanation of the cutoff mechanism, was questioned in Refs. [11,17,18]. Some of these works are based on a field theory developed in Ref. [11] where the dynamical action is invariant under a set of linear

time-reversal transformations. This formulation allows one to have a full set of fluctuation-dissipation relations (FDR) relating linearly correlation functions to response functions. The field theory of DM has only a limited number of linear FDR and some relations hold only in the hydrodynamic limit. This is one of the points on which the conclusion by DM on the absence of the sharp ENE transition was questioned. The field theory with linear FDR was later improved [14] for the case of interacting Brownian particles satisfying the Dean-Kawasaki equation [19,20], where the standard MCT result was recovered at the one-loop order of perturbation theory. This improved method was then applied to the FNH [17] with results indicating a sharp ENE transition at the one-loop order with the nonergodicity parameter satisfying the standard MCT result. In response to these developments, DM reexamined their work and showed [21] in a nonperturbative analysis without resorting to the hydrodynamic limit that the sharp ENE transition is not present in the FNH after all. This conclusion is also supported by the recent direct numerical integration of the generalized Langevin equations of the FNH [22].

It is somewhat puzzling to have completely different results from the two field theoretic approaches of the same model. In this respect, it might be desirable to have a simpler setting in which one can compare these two field theoretical approaches and study where the difference originates. The field theoretical treatment of FNH is complicated by many factors including the presence of many dynamical variables. This is especially the case for the field theory with linear FDR. In this paper, we present a simple toy model of FNH, which can shed some light on the issues described above concerning the field theoretic approaches to the FNH. In this toy model, there is no spatial dependence in the dynamical field variables which consist simply of a density-like variable and a single-component momentum-like variable. We develop the two different field theories of the toy model,

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[*]jhyeo@konkuk.ac.kr

namely the original DM-type field theory and the one with linear FDR. We show explicitly that, at the one-loop order of perturbation expansion, a sharp ENE-type transition does not occur in the DM-type field theory. On the other hand, the field theory with linear FDR shows a sharp transition at the one-loop order. By comparing the two field theories, we find that the major difference between the two formulations lies in the way of treating the density nonlinearities present in FNH within the renormalized perturbation theory. In particular, the field theory with linear FDR results in a dynamical action which contains nonpolynomial functions of field variables in contrast to the DM field theory. This implies that, when the renormalized perturbation theory is performed at a given order of the loop expansion, the two field theories end up with treating the density nonlinearities in a different way, since the field theory with linear FDR involves truncating the nonpolynomial functions.

Although there is only a limited number of linear FDR, the DM field theory at a given order of the loop expansion can be regarded as a well-defined self-consistent theory among the correlation and response functions satisfying a set of coupled self-consistent equations. In this paper, we construct the set of coupled equations at the one-loop order for the DM field theoretic approach to the toy model and study them numerically. Since the simple nature of the toy model reduces the number of independent correlation and response functions, we were able to solve these equations numerically.

In the next section, we present our toy model and construct the field theories following the DM prescription and the method involving linear FDR, respectively. In Sec. III, we study the time evolution of the correlation functions using the Schwinger-Dyson equations for both field theoretic formulations. We then analyze the time evolution equations for the correlation function corresponding to the density autocorrelation function of the FNH at the one-loop order for the possible existence of a sharp ENE-type transition. In Sec. IV, we present a set of coupled equations for the correlation and response functions in the DM field theory at the one-loop order and their numerical solutions. In the final section, we summarize our results with discussion.

## II. MODEL

Our model is a zero-dimensional version of the FNH of compressible fluids developed by Das and Mazenko [8]. We introduce as our dynamical variables a time-dependent density-like variable $a(t)$ and a single-component momentum-like variable $b(t)$ without any spatial degrees of freedom. In order to construct the equations motion for these variables, we introduce the effective free energy $F$ where the equilibrium distribution for the system at temperature $T$ is given by $\exp(-F/T)$. The free energy can be written as $F=F_K+F_U$ where $F_K[a,b]$ is the kinetic energy and the potential energy part $F_U[a]$ is assumed to depend only on $a$. We take the usual form for the kinetic part, that is $F_K=\frac{b^2}{2a}$. The nonlinearity in the form of $1/a$ plays an important role in the following discussion.

Kim and Kawasaki [23] introduced a similar zero-dimensional toy model of the FNH to the present one by

incorporating the multicomponent density-like and momentum-like variables. However, in addition to having multicomponent fields, their model differs from ours in a fundamental way. Their free energy is quadratic both in the density-like and the momentum-like variables without the $1/a$ nonlinearity which is present in the FNH of compressible fluids. In the present toy model, therefore, the actual form of the equations of motion will be different from those in Ref. [23], but the derivation of the equations from the free energy can be performed in the same way. In our model, the equation of motion for the variable $a(t)$ takes the form of a zero-dimensional version of the continuity equation, namely

$$\dot{a}(t) + Jb(t) = 0 \qquad (1)$$

for some constant $J$. This can be regarded as the reversible dynamics for $a(t)$ which can be derived from

$$\dot{a}(t) = Q_{ab}\frac{\partial F}{\partial b} - T\frac{\partial Q_{ab}}{\partial b}, \qquad (2)$$

with $Q_{ab}=-Ja$ playing the role of the Poisson bracket. The equation of motion for $b(t)$ has the dissipative part described by the coefficient $\Gamma$ in addition to the reversible part as follows:

$$\dot{b}(t) = Q_{ba}\frac{\partial F}{\partial a} - T\frac{\partial Q_{ba}}{\partial a} - \Gamma\frac{\partial F}{\partial b} + \theta(t), \qquad (3)$$

where $Q_{ba}=-Q_{ab}$ and the Gaussian white noise $\theta$ has the variance $\langle\theta(t)\theta(t')\rangle=2\Gamma T\delta(t-t')$. In the present work, we take the simple quadratic form for the potential energy part $F_U$ of the effective free energy $F$ as

$$F_U[a] = \frac{A}{2}(\delta a)^2 \qquad (4)$$

with the fluctuation $\delta a=a-a_0$ and the average value $a_0$. We thus have the equation of motion for $b(t)$ as

$$\dot{b}(t) + J\left(\frac{b^2}{2a}\right) - JAa(\delta a) + TJ + \Gamma\left(\frac{b}{a}\right) = \theta, \qquad (5)$$

We can easily verify that the equilibrium stationary distribution corresponding to the above equations for $a$ and $b$ is proportional to $\exp(-F/T)$. The corresponding Fokker-Planck equation for the probability distribution $P(a,b,t)$ is given by $\partial_t P = \mathcal{L}P$ where the Fokker-Planck operator is given by $\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2$ with

$$\mathcal{L}_1 = \frac{\partial}{\partial b}\Gamma\left(T\frac{\partial}{\partial b} + \frac{b}{a}\right) \qquad (6)$$

$$\mathcal{L}_2 = \frac{\partial}{\partial a}(Jb) + \frac{\partial}{\partial b}\left[J\left(\frac{b^2}{2a}\right) - JAa(\delta a) + TJ\right]. \qquad (7)$$

It is straightforward to show that $P \sim \exp(-F/T)$ satisfies $\mathcal{L}P=0$.

One can develop a field theory from the above Langevin equations by using the standard Martin-Siggia-Rose (MSR) formalism [24]. In the MSR procedure, the hatted fields $\hat{a}$ and $\hat{b}$ are introduced to enforce the equations of motion for $a$

and $b$, respectively. We present in the following subsections two different field theoretical approaches to this model and compare the outcomes of both approaches concerning the existence of an ENE transition. The first one is the original Das and Mazenko approach [8], where the $1/a$ nonlinearities in the model are taken care of in a simple way by the introduction of a single additional auxiliary field. The second approach due to Refs. [11,14,17] incorporates the linear time-reversal symmetry into the dynamical action resulting in a set of linear FDR. In order to do that, more auxiliary fields will have to be introduced.

### A. Das-Mazenko approach

In the DM approach, an auxiliary velocity-like field $c(t)$ is presented such that the condition $b(t)=a(t)c(t)$ is enforced through a delta function

$$1 = \int \mathcal{D}c(t)\,\delta[a(t)c(t)-b(t)]$$

$$= \int \mathcal{D}c(t) \int \mathcal{D}\hat{c}(t)\exp\{-i\hat{c}(t)[b(t)-a(t)c(t)]\}. \quad (8)$$

The first equality holds up to a Jacobian. This Jacobian was shown in Ref. [25] to have no effect on the correlation and response functions and will be neglected in the following analysis. Using this identity, we obtain the generating functional $Z$ as a functional integral over the fields $\psi_i(t) = \delta a(t), b(t), c(t)$ and $\hat{\psi}_i(t) = \hat{a}(t), \hat{b}(t), \hat{c}(t)$. We can write $Z_{\mathrm{DM}} = \int \Pi_i \mathcal{D}\psi_i \mathcal{D}\hat{\psi}_i \exp(-S_{\mathrm{DM}}[\psi,\hat{\psi}])$, where

$$S_{\mathrm{DM}} = \int dt \left\{ \Gamma T\hat{b}^2(t) + i\hat{a}(t)[\dot{a}(t)+Jb(t)] \right.$$

$$+ i\hat{b}(t)\left\{ \dot{b}(t) - JAa_0\delta a(t) + \Gamma c(t) + TJ + \frac{Ja_0}{2}[c(t)]^2 \right.$$

$$\left. + \frac{J}{2}\delta a(t)[c(t)]^2 - JA[\delta a(t)]^2 \right\} + i\hat{c}(t)\{b(t)-a_0 c(t)$$

$$\left. - \delta a(t)c(t)\} \right\}. \quad (9)$$

We will use $\Psi(t)$ to represent any one of the six variables $\{\psi_i, \hat{\psi}_i\}$ in our model and denote the two-point correlation function between arbitrary two variables $\Psi(t)$ and $\Psi'(t')$ by

$$G_{\Psi\Psi'}(t-t') = \langle \Psi(t)\Psi'(t') \rangle. \quad (10)$$

(For the subscripts of $G$, we will use $a$ instead of $\delta a$ for simplicity.) Note that among the correlation functions those between two hatted variables vanish due to causality, that is $G_{\hat{\psi}_i\hat{\psi}_j}=0$. It follows that $iTJ\hat{b}(t)$ term in the action Eq. (9) has no effect on the correlation functions and will be neglected in the following. The causality also requires that $G_{\psi_i\hat{\psi}_j}(t)=0$ for $t<0$.

We can easily establish some nonperturbative relations among the correlation functions which will be useful in later discussion. If we use $\Psi(t')=\psi_i(t')=\delta a(t'), b(t')$ and $c(t')$ in the identity

$$0 = \int \prod_i \mathcal{D}\psi_i \mathcal{D}\hat{\psi}_i \frac{\delta}{\delta\hat{a}(t)}[\Psi(t')\exp(-S_{\mathrm{DM}})], \quad (11)$$

we obtain

$$-i\frac{\partial}{\partial t}\langle \psi_i(t')a(t) \rangle - iJ\langle \psi_i(t')b(t) \rangle = 0,$$

which can be rewritten as

$$\dot{G}_{a\psi}(t) + JG_{b\psi}(t) = 0. \quad (12)$$

On the other hand, if $\Psi(t')=\hat{\psi}(t')=\hat{a}(t'), \hat{b}(t')$ and $\hat{c}(t')$ are used in Eq. (11), we have

$$\dot{G}_{a\hat{\psi}}(t) + JG_{b\hat{\psi}}(t) = -i\delta_{\hat{\psi}\hat{a}}\delta(t). \quad (13)$$

Note that Eqs. (12) and (13) are direct consequences of the zero-dimensional version of the mass conservation law given by Eq. (1) in the DM approach. Note also that the bare correlation functions obtained from the dynamical action $S_{\mathrm{DM}}^{(0)}$ which contains only the quadratic terms in the field variables also satisfy these relations. Therefore, when we develop the perturbation theory for the DM approach in the next section, these equations will hold order-by-order in the perturbation expansion.

In the DM approach, only a limited number of FDR exist that relate linearly the correlation functions to response functions. Assuming the time-reversal properties of the fields as $a(-t)=a(t)$, $b(-t)=-b(t)$, and $c(-t)=-c(t)$, we can derive the FDR for $\psi=a$, $b$, and $c$ as

$$G_{\psi\hat{b}}(t) = -\frac{i}{T}\Theta(t)G_{\psi c}(t), \quad (14)$$

where $\Theta(t)=1$ for $t>0$ and vanishes for $t<0$. The detailed derivation of the FDR closely follows the one given in Ref. [8]. Since $\hat{\psi}_i$ is a real field, we can show that the correlation function between unhatted and hatted variables is a pure imaginary number, that is

$$G_{\psi_i\hat{\psi}_j}^*(t) = -G_{\psi_i\hat{\psi}_j}(t). \quad (15)$$

### B. Field theory with linear FDR

We apply the field theoretical approach developed in Refs. [11,14,17] to our toy model given by the dynamic Eqs. (1) and (5). Among these methods, we will follow closely the improved procedure described in Ref. [17]. In order to do that, we present two auxiliary fields $\eta(t)$ and $\theta(t)$ defined by

$$\eta = \frac{\partial F}{\partial b} - \frac{b}{a_0} = -\frac{b}{a_0}\sum_{k=1}^{\infty}(-1)^k\left(\frac{\delta a}{a_0}\right)^k \equiv f_\eta(\delta a, b) \quad (16)$$

$$\theta = \frac{\partial F}{\partial a} - A(\delta a) = -\frac{b^2}{2a_0^2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\delta a}{a_0}\right)^k \equiv f_\theta(\delta a, b). \quad (17)$$

Note that these are nonpolynomial functions of the main dynamical variables $\delta a$ and $b$. Presenting the hatted counter-

parts $\hat{\eta}(t)$ and $\hat{\theta}(t)$ to enforce these definitions for the variables, we can construct the generating functional as $Z_{\text{FDR}} = \int \Pi_i \mathcal{D}\phi_i \mathcal{D}\hat{\phi}_i \exp(-S_{\text{FDR}}[\phi,\hat{\phi}])$, where $\phi_i(t) = \delta a(t), b(t), \eta(t), \theta(t)$ and $\hat{\phi}_i(t) = \hat{a}(t), \hat{b}(t), \hat{\eta}(t), \hat{\theta}(t)$. Similarly to the DM case, we will use $\Phi(t)$ to represent one of the eight variables $\{\phi_i, \hat{\phi}_i\}$. The dynamical action in this case can be written as a sum of the Gaussian and the nonlinear parts as $S_{\text{FDR}} = S_{\text{FDR}}^{(0)} + S_{\text{FDR}}^{(1)}$, where

$$S_{\text{FDR}}^{(0)} = \int dt \left[ \Gamma T \hat{b}^2(t) + i\hat{a}(t)\{\dot{a}(t) + Jb(t) + Ja_0\eta(t)\} + i\hat{b}(t) \right.$$
$$\times \left\{ \dot{b}(t) - JAa_0\delta a(t) - Ja_0\theta(t) + \Gamma\eta(t) + \frac{\Gamma}{a_0}b(t) \right\}$$
$$\left. + i\hat{\eta}(t)\eta(t) + i\hat{\theta}(t)\theta(t) \right], \tag{18}$$

and

$$S_{\text{FDR}}^{(1)} = \int dt \left\{ i\hat{a}(t)\left[ J\delta a(t)\eta(t) + \frac{J}{a_0}\delta a(t)b(t) \right] - i\hat{b}(t) \right.$$
$$\times \{ J\delta a(t)\theta(t) + JA[\delta a(t)]^2 \} - i\hat{\eta}(t)f_\eta(\delta a, b)$$
$$\left. - i\hat{\theta}(t)f_\theta(\delta a, b) \right\}. \tag{19}$$

Note that we have used the identity

$$a_0\eta(t) + \delta a(t)\eta(t) + \frac{1}{a_0}\delta a(t)b(t) = 0, \tag{20}$$

which follows directly from the definition of $\eta(t)$, Eq. (16). The above actions $S_{\text{FDR}}^{(0)}$ and $S_{\text{FDR}}^{(1)}$ are separately invariant under the time-reversal transformations given by

$$\delta a(-t) = \delta a(t), \quad b(-t) = -b(t),$$

$$\eta(-t) = -\eta(t), \quad \theta(-t) = \theta(t),$$

$$\hat{a}(-t) = -\hat{a}(t) - \frac{i}{T}\theta(t) - i\frac{A}{T}\delta a(t),$$

$$\hat{b}(-t) = \hat{b}(t) + \frac{i}{T}\eta(t) + \frac{i}{a_0 T}b(t),$$

$$\hat{\eta}(-t) = -\hat{\eta}(t) + \frac{i}{T}\dot{b}(t),$$

$$\hat{\theta}(-t) = \hat{\theta}(t) - \frac{i}{T}\dot{a}(t). \tag{21}$$

Applying the time-reversal invariance on the various correlation functions by following the procedures described in Refs. [11,14,17], we obtain a set of FDR. Here we only list those which are relevant to the discussion in the next section. We have for $\phi = a, b, \eta,$ and $\theta$

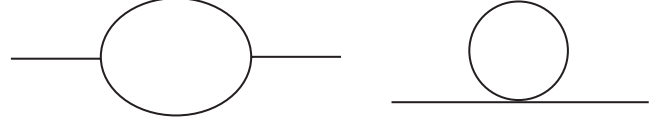$$G_{\phi\hat{a}}(t) = -\frac{i}{T}\Theta(t)[AG_{\phi a}(t) + G_{\phi\theta}(t)],$$



FIG. 1. The one-loop diagrams contributing to the self-energy $\Sigma$.

$$G_{\phi\hat{b}}(t) = -\frac{i}{T}\Theta(t)\left[ \frac{1}{a_0}G_{\phi b}(t) + G_{\phi\eta}(t) \right],$$

$$G_{\phi\hat{\eta}}(t) = -\frac{i}{T}\Theta(t)\dot{G}_{\phi b}(t),$$

$$G_{\phi\hat{\theta}}(t) = -\frac{i}{T}\Theta(t)\dot{G}_{\phi a}(t). \tag{22}$$

## III. RENORMALIZED PERTURBATION THEORY: ONE-LOOP ORDER

In this section, we develop self-consistent renormalized perturbation theories for the two field theoretic approaches introduced in the previous section. We then focus on the time evolution of two-point correlation functions using the Schwinger-Dyson (SD) equation. In particular, we study the $t \to \infty$ limit of $G_{aa}(t)$, which corresponds to the density autocorrelation function in FNH, to explore the possibility of an ENE transition. The formal development of the self-consistent perturbation theory can be found in Refs. [14,26,27]. The SD equation defines the self-energy $\Sigma$ through its relation to the propagator (the two-point correlation function) $\mathbf{G}$. It is given symbolically by

$$\mathbf{G}^{-1} = \mathbf{G}_0^{-1} - \Sigma, \tag{23}$$

where the subscript 0 refers to the bare quantity obtained by keeping only the Gaussian terms in the action. The self-energy is obtained by differentiating the so-called two-particle irreducible vertex function $\Gamma_{2\text{PI}}[\mathbf{G}]$ with respect to the propagator $\mathbf{G}$. At the one-loop order of the loop expansion of $\Gamma_{2\text{PI}}$, there are only two kinds of diagrams for the self-energy which are relevant to the two field theoretical approaches studied in the previous section. These are shown in Fig. 1. The diagram on the right hand side is needed to take account of the quartic nonlinear terms in the dynamic action. The contribution from this diagram to the self-energy as a function of time $t$ will be proportional to $\delta(t)$, and therefore does not play a significant role in the discussion of the $t \to \infty$ limit given in this section. However, when we solve numerically the full self-consistent equations among the correlation functions in the DM approach in the next section, the contributions from all the diagrams must be included.

### A. Das-Mazenko approach

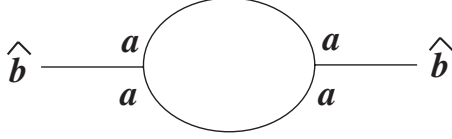The SD equation for the DM field theory between arbitrary two fields $\Psi$ and $\Psi'$ is given by

FIG. 2. The one-loop diagram for $\Sigma_{\hat{b}\hat{b}}$ that may have a nonvanishing contribution in the $t \to \infty$ limit.

$$\delta_{\Psi\Psi'}\delta(t-t') = \sum_{\Psi''} \int dt'' \{ [G_0^{-1}]_{\Psi\Psi''}(t-t') G_{\Psi''\Psi'}(t''-t')$$

$$- \Sigma_{\Psi\Psi''}(t-t'') G_{\Psi''\Psi'}(t''-t') \}. \qquad (24)$$

We note that the causality requirement for the self-energy reads $\Sigma_{\psi_i\psi_j} = 0$ for $t < 0$ and $\Sigma_{\psi_i\psi_j}(t) = 0$ for all $t$. Similarly to the propagators, the self-energy between unhatted and hatted variables is pure imaginary. Since there is no nonlinear term containing $b(t)$ in Eq. (9), the self-energies involving $b$ must vanish. From this feature of the DM field theory, we can derive nonperturbative relations among correlation functions as

$$JG_{a\hat{a}}(t) + \dot{G}_{a\hat{b}}(t) + G_{a\hat{c}}(t) = 0, \qquad (25)$$

$$JG_{c\hat{a}}(t) + \dot{G}_{c\hat{b}}(t) + G_{c\hat{c}}(t) = 0 \qquad (26)$$

from the $(\Psi,\Psi') = (b,a)$ and $(b,c)$ component of the SD equation, respectively.

In order to study the time evolution of $G_{aa}(t)$, we look at the $(\Psi,\Psi') = (\hat{b},a)$ component, which yields

$$\dot{G}_{ba}(t) + \Gamma G_{ca}(t) - JA a_0 G_{aa}(t) - 2i\Gamma T G_{\hat{b}a}(t) = F_{\hat{b}a}(t), \qquad (27)$$

where

$$F_{\hat{b}a}(t) = -i \int_{-\infty}^{t} ds \{ \Sigma_{\hat{b}a}(t-s) G_{aa}(s) + \Sigma_{\hat{b}c}(t-s) G_{ca}(s) \}$$

$$- i \int_{-\infty}^{0} ds \{ \Sigma_{\hat{b}\hat{b}}(t-s) G_{\hat{b}a}(s) + \Sigma_{\hat{b}\hat{c}}(t-s) G_{\hat{c}a}(s) \}. \qquad (28)$$

Using Eqs. (12) and (14), we can rewrite this equation for $t > 0$ as

$$\ddot{G}_{aa}(t) - J\Gamma G_{ca}(t) + J^2 A a_0 G_{aa}(t) = -JF_{\hat{b}a}(t). \qquad (29)$$

Now in order to investigate the possible ENE transition in this model, we consider the $t \to \infty$ limit. Let us assume that all the other correlation functions except $G_{aa}(t)$ vanish in the $t \to \infty$ limit. To the one-loop order of the perturbation expansion, only the self-energy $\Sigma_{\hat{b}\hat{b}}(t)$ can be nonvanishing in the $t \to \infty$ limit due to the diagram shown in Fig. 2, since it is proportional to $[G_{aa}(t)]^2$. Therefore, the nonvanishing contributions to $F_{\hat{b}a}(\infty)$ come from the first and the third terms on the right-hand side of Eq. (28). Among the terms on the left hand side of Eq. (29) only the third term is nonvanishing in this limit. Using Eq. (14), we therefore have

$$JA a_0 G_{aa}(\infty) = \sigma G_{aa}(\infty) + \frac{1}{T}\Sigma_{\hat{b}\hat{b}}(\infty) \int_0^{\infty} ds G_{ac}(s), \quad (30)$$

where $\sigma = \int_0^{\infty} ds i\Sigma_{\hat{b}a}(s)$ is a finite real number. The relation between $G_{ac}(t)$ and $G_{aa}(t)$ can be obtained from the $(\Psi,\Psi') = (\hat{c},a)$ component of the SD equation, which is given by

$$-\frac{1}{J}\dot{G}_{aa}(t) + a_0 G_{ac}(t) = F_{\hat{c}a}(t), \qquad (31)$$

where

$$F_{\hat{c}a}(t) = -i \int_{-\infty}^{t} ds \{ \Sigma_{\hat{c}a}(t-s) G_{aa}(s) + \Sigma_{\hat{c}c}(t-s) G_{ca}(s) \}$$

$$- i \int_{-\infty}^{0} ds \{ \Sigma_{\hat{c}\hat{b}}(t-s) G_{\hat{b}a}(s) + \Sigma_{\hat{c}\hat{c}}(t-s) G_{\hat{c}a}(s) \}. \qquad (32)$$

Now let us suppose that $F_{\hat{c}a}(t)$ can be taken to zero for some reason, then, by inserting the expression for $G_{ac}(t)$ from Eq. (31) into Eq. (30), we obtain an equation for $G_{aa}(\infty)$, which may have a nonvanishing solution signaling an ENE transition. This is essentially what happens in the field theory with linear FDR as we will see in the next subsection. In the DM field theory, however, the presence of the first term on the right hand side of Eq. (32) spoils this scenario. In fact, as $t \to \infty$, $F_{\hat{c}a}(t)$ approaches

$$G_{aa}(\infty) \int_0^{\infty} ds(-i)\Sigma_{\hat{c}a}(s),$$

which is nonvanishing by assumption. Then the integral in Eq. (30) becomes ill-defined and we are forced to abandon the assumption of the nonzero $G_{aa}(\infty)$. This finding is consistent with the recent nonpertubative proof by Das and Mazenko [21] that the FNH will full spatial dependence does not support a sharp ENE transition. We note that the absence of the ENE transition in our model is directly related to the presence of the self-energy $\Sigma_{\hat{c}a}$ in our model. This is also similar to the result of Ref. [21], where the self-energy coupling the hatted velocity field and the density field plays a crucial role in removing the sharp transition.

### B. Field theory with linear FDR

The SD equation in this case is given similarly to Eq. (24) but now with the component $\Phi$ taking eight different field variables, $\{\phi_i, \hat{\phi}_i\}$. So there are a lot more equations to consider in this approach. The self-energies are, however, related to each other through many FDR, which can be obtained by applying the time-reversal invariance to the SD equation. Here we list only the relevant FDR among the self-energies to our discussion. For $\hat{\phi} = \hat{a}$, $\hat{b}$, $\hat{\eta}$, or $\hat{\theta}$, we have

$$\Sigma_{\hat{\phi}a}(t) = \frac{i}{T}\Theta(t)[A\Sigma_{\hat{\phi}\hat{a}}(t) - \dot{\Sigma}_{\hat{\phi}\hat{\theta}}(t)],$$

$$\Sigma_{\hat{\phi}b}(t) = \frac{i}{T}\Theta(t)\left[\frac{1}{a_0}\Sigma_{\hat{\phi}\hat{b}}(t) - \dot{\Sigma}_{\hat{\phi}\hat{\eta}}(t)\right],$$

$$\Sigma_{\hat{\phi}\eta}(t) = \frac{i}{T}\Theta(t)\Sigma_{\hat{\phi}\hat{b}}(t),$$

$$\Sigma_{\hat{\phi}\theta}(t) = \frac{i}{T}\Theta(t)\Sigma_{\hat{\phi}\hat{a}}(t). \tag{33}$$

As in the DM approach, we study the time evolution of $G_{aa}(t)$ and its infinite-time limit for the possible ENE transition. From the $(\Phi,\Phi')=(\hat{b},a)$ component of the SD equation, we have

$$\dot{G}_{ba}(t) + \frac{\Gamma}{a_0}G_{ba}(t) - JAa_0 G_{aa}(t) - 2i\Gamma T G_{\hat{b}a}(t) + \Gamma G_{\eta a}(t)$$

$$- Ja_0 G_{\theta a}(t) = \widetilde{F}_{\hat{b}a}(t), \tag{34}$$

where

$$\widetilde{F}_{\hat{b}a}(t) = -i\int_{-\infty}^{t} ds\sum_{\phi'}\Sigma_{\hat{b}\phi'}(t-s)G_{\phi'a}(s)$$

$$- i\int_{-\infty}^{0} ds\sum_{\hat{\phi}'}\Sigma_{\hat{b}\hat{\phi}'}(t-s)G_{\hat{\phi}'a}(s). \tag{35}$$

Using the FDR in Eqs. (22) and (33), we can rewrite the above quantity (multiplied by $T$) as

$$T\widetilde{F}_{\hat{b}a}(t) = [\Sigma_{\hat{b}\hat{a}}\otimes(AG_{aa} + G_{\theta a})](t)$$

$$+ \left[\Sigma_{\hat{b}\hat{b}}\otimes\left(\frac{1}{a_0}G_{ba} + G_{\eta a}\right)\right](t)$$

$$- [\Sigma_{\hat{b}\hat{\eta}}\otimes\dot{G}_{ba}](t) - [\Sigma_{\hat{b}\hat{\theta}}\otimes\dot{G}_{aa}](t), \tag{36}$$

where the convolution between two function $f(t)$ and $g(t)$ is defined by

$$[f\otimes g](t) \equiv \int_{0}^{t} ds f(t-s)g(s). \tag{37}$$

As in the DM approach, from the simple form of the equation for $a(t)$ and Eq. (20), we can derive a nonperturbative relation, namely,

$$JG_{ba}(t) + \dot{G}_{aa}(t) = 0. \tag{38}$$

We now consider the $t\to\infty$ limit. If we assume as in the previous subsection that all the other correlation functions except $G_{aa}(t)$ vanishes in this limit, then to the one-loop order, the only nonvanishing diagram in this limit is again the one in Fig. 2. The contribution from this diagram to $\Sigma_{\hat{b}\hat{b}}(t)$ is $-2J^2A^2G_{aa}^2(t)$. Taking the $t\to\infty$ limit in Eq. (34) and using Eqs. (36) and (38), we obtain

$$- JAa_0 G_{aa}(\infty) = A\widetilde{\sigma}G_{aa}(\infty) - \frac{1}{JTa_0}\Sigma_{\hat{b}\hat{b}}(\infty)\int_{0}^{\infty} ds\dot{G}_{aa}(s), \tag{39}$$

where

$$\widetilde{\sigma} = \frac{1}{T}\int_{0}^{\infty} ds\Sigma_{\hat{b}\hat{a}}(s). \tag{40}$$

We therefore have

$$G_{aa}(\infty) = \frac{\Sigma_{\hat{b}\hat{b}}(\infty)}{J^2\widetilde{A}Ta_0^2}[G_{aa}(\infty) - G_{aa}(0)], \tag{41}$$

where $\widetilde{A} = A(1+\widetilde{\sigma}/(Ja_0))$. Defining the nonergodicity parameter by $f = G_{aa}(\infty)/G_{aa}(0)$, we have

$$\frac{f}{1-f} = c_2 f^2, \tag{42}$$

where

$$c_2 = \frac{2A^2[G_{aa}(0)]^2}{T\widetilde{A}a_0^2} \tag{43}$$

is a dimensionless quantity. This is exactly the standard MCT equation for the nonergodicity parameter in the so-called schematic model of the standard MCT [5,6]. The nonergodic solution $f>0$ exists when $c_2>4$.

The origin of the difference between the results of the two field theoretic approaches on the existence of an ENE transition can be traced back to the terms that are multiplied by the self-energy $\Sigma_{\hat{b}\hat{b}}$. In both cases, this term is expressed as a time integral of a correlation function. In the field theory with linear FDR, this correlation function is proportional to a total time derivative of $G_{aa}(t)$. From this, a well-defined equation like Eq. (42) follows for the nonergodicity parameter. In the DM approach, however, the correlation function in the integrand is not a total time derivative of $G_{aa}(t)$, but contains an extra contribution from the self-energy coupling the hatted auxiliary field $\hat{c}$ and the density-like field $a$. On another level, we can understand that the difference comes from the fact that, for the one-loop calculation in the field theory with linear FDR, only the first-order terms are used among those in the expression for the fields $\eta$ and $\theta$ in Eqs. (16) and (17). This truncation of the nonpolynomial action in the perturbation expansion does not occur in the DM field theory. This suggests that the cutoff of a sharp transition that the DM approach exhibits at the one loop order is a kind of nonpertubative information that only an infinite resummation in the field theory with linear FDR would have an access to.

## IV. NUMERICAL CALCULATIONS FOR THE DAS-MAZENKO FIELD THEORY

In order to study not just the nonergodicity parameter, but the full time evolution of $G_{aa}(t)$, it is desirable to have a single equation for the correlation function that accounts for the density feedback mechanism. In the original FNH in the

DM field theoretic approach [8], this was achieved only in the hydrodynamic limit. It is, however, difficult even in the simple toy model to find a time evolution equation only for $G_{aa}(t)$. In this section, we show that despite the lack of a complete set of linear FDR, the DM field theoretical formulation presents a well-defined theory at one-loop order. We do this by constructing a closed set of equations for all the correlation functions involved and by solving them for $G_{aa}(t)$ numerically. Because of the simple nature of the toy model, especially of Eqs. (12)–(14), (25), and (26), we only have five independent correlation functions in the DM field theory. We choose them to be $G_{aa}(t)$, $G_{a\hat{b}}(t)$, $G_{a\hat{c}}(t)$, $G_{c\hat{b}}(t)$, and $G_{c\hat{c}}(t)$. All the other correlation functions can be expressed in terms of these five functions.

If we define the Fourier transform by $\widetilde{f}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} f(t)$ for an arbitrary function $f(t)$, the Fourier transforms of these five correlation functions can be written as

$$\widetilde{G}_{aa}(\omega) = \frac{J^2}{|D(\omega)|^2} \{ |a_R(\omega)|^2 [2\Gamma T - \widetilde{\Sigma}_{\hat{b}\hat{b}}(\omega)] $$
$$- 2 \, \mathrm{Re}[a_R(\omega)\Gamma_R^*(\omega)\widetilde{\Sigma}_{\hat{b}\hat{c}}(\omega)] - |\Gamma_R(\omega)|^2 \widetilde{\Sigma}_{\hat{c}\hat{c}}(\omega) \},$$
$$(44)$$

$$\widetilde{G}_{a\hat{b}}(\omega) = \frac{-iJa_R(\omega)}{D(\omega)}, \tag{45}$$

$$\widetilde{G}_{a\hat{c}}(\omega) = \frac{-iJ\Gamma_R(\omega)}{D(\omega)}, \tag{46}$$

$$\widetilde{G}_{c\hat{b}}(\omega) = \frac{\omega + J\widetilde{\Sigma}_{\hat{c}a}(\omega)}{D(\omega)}, \tag{47}$$

$$\widetilde{G}_{c\hat{c}}(\omega) = \frac{i\omega^2 - iJK_R(\omega)}{D(\omega)}, \tag{48}$$

where Re denotes the real part and

$$a_R(\omega) = a_0 - i\widetilde{\Sigma}_{\hat{c}c}(\omega), \tag{49}$$

$$\Gamma_R(\omega) = \Gamma + i\widetilde{\Sigma}_{\hat{b}c}(\omega), \tag{50}$$

$$K_R(\omega) = JAa_0 - i\widetilde{\Sigma}_{\hat{b}a}(\omega), \tag{51}$$

$$D(\omega) = a_R(\omega)[\omega^2 - JK_R(\omega)] + i\Gamma_R(\omega)[\omega + i\widetilde{\Sigma}_{\hat{c}a}(\omega)]. \tag{52}$$

Note that the Fourier transforms of seven self-energies are involved in the above equations, which are $\widetilde{\Sigma}_{\hat{b}a}(\omega)$, $\widetilde{\Sigma}_{\hat{b}c}(\omega)$, $\widetilde{\Sigma}_{\hat{c}a}(\omega)$, $\widetilde{\Sigma}_{\hat{c}c}(\omega)$, $\widetilde{\Sigma}_{\hat{b}\hat{b}}(\omega)$, $\widetilde{\Sigma}_{\hat{b}\hat{c}}(\omega)$, and $\widetilde{\Sigma}_{\hat{c}\hat{c}}(\omega)$. Note also that other correlation functions can be written in terms of the five given above. For example, we have $\widetilde{G}_{ba}(\omega) = (i\omega/J)\widetilde{G}_{aa}(\omega)$, which is nothing but the mass conservation law given in Eq. (12). When one evaluates the self-energies order–by–order in the perturbation expansion, one can easily see from Eq. (44)

and this explicit relationship between the two correlation functions that the conservation law holds automatically.

At the one-loop order of the perturbation theory, these seven self-energies are given by functions of the five independent correlation functions thus yielding a closed set of equations. It is more convenient to present the one-loop self-energies in the time domain. (The detailed derivation of the one-loop self-energies in terms of the correlations functions will be given elsewhere [28].) They are given by

$$\Sigma_{\hat{b}a}(t) = -2ia_0 J^2 AT G_{a\hat{b}}(t)G_{c\hat{b}}(t) + ia_0 JT G_{c\hat{b}}(t)G_{c\hat{c}}(t)$$
$$- 4J^2 A^2 G_{aa}(t)G_{a\hat{b}} - 2iJATG_{a\hat{b}}(t)G_{a\hat{c}}(t),$$

$$+ \frac{1}{2}JTG_{c\hat{b}}(0^+)\delta(t), \tag{53}$$

$$\Sigma_{\hat{b}c}(t) = -ia_0^2 J^2 T [G_{c\hat{b}}(t)]^2 - ia_0 JT G_{a\hat{b}}(t)G_{c\hat{c}}(t)$$
$$+ 2ia_0 J^2 AT [G_{a\hat{b}}(t)]^2 - 2JAG_{aa}(t)G_{a\hat{c}}(t), \tag{54}$$

$$\Sigma_{\hat{c}a}(t) = -2JAG_{aa}(t)G_{c\hat{b}}(t) + 2iJAT [G_{a\hat{b}}(t)]^2$$
$$- iTG_{a\hat{b}}(t)G_{c\hat{c}}(t) - iTG_{a\hat{c}}(t)G_{c\hat{b}}(t), \tag{55}$$

$$\Sigma_{\hat{c}c}(t) = 2ia_0 JT G_{a\hat{b}}(t)G_{c\hat{b}}(t) + iTG_{a\hat{b}}(t)G_{a\hat{c}}(t) - G_{aa}(t)G_{c\hat{c}}(t), \tag{56}$$

$$\Sigma_{\hat{b}\hat{b}}(t) = -2J^2 A^2 [G_{aa}(t)]^2 - 2a_0 J^2 AT^2 [G_{a\hat{b}}(t)]^2$$
$$+ \frac{1}{2}a_0^2 J^2 T^2 [G_{c\hat{b}}(t)]^2, \tag{57}$$

$$\Sigma_{\hat{b}\hat{c}}(t) = -2iJAT G_{aa}(t)G_{a\hat{b}}(t) + a_0 JT^2 G_{a\hat{b}}(t)G_{c\hat{b}}(t), \tag{58}$$

$$\Sigma_{\hat{c}\hat{c}}(t) = -iTG_{aa}(t)G_{c\hat{b}}(t) - T^2 [G_{a\hat{b}}(t)]^2. \tag{59}$$

Note that these expressions are valid only for $t > 0$. The self-energies in Eqs. (53)–(56) vanish for $t < 0$ due to causality. The remaining self-energies satisfy $\Sigma_{\hat{b}\hat{b}}(-t) = \Sigma_{\hat{b}\hat{b}}(t)$, $\Sigma_{\hat{c}\hat{c}}(-t) = \Sigma_{\hat{c}\hat{c}}(t)$, and $\Sigma_{\hat{b}\hat{c}}(-t) = -\Sigma_{\hat{b}\hat{c}}(t)$. The last term in Eq. (53) is the contribution from the diagram on the right hand side of Fig. 1.

There is one point that requires caution in performing a numerical calculation on these types of self-consistent equations. We note that $\widetilde{G}_{c\hat{c}}(\omega)$ does not decay to zero as $\omega \to \infty$. This suggests that there is a delta-function singularity in $G_{c\hat{c}}(t)$ at short time, which has to be treated separately in a numerical calculation. We write $G_{c\hat{c}}(t) = i\alpha\delta(t)+$ regular terms for some real constant $\alpha$. If we denote by $f'$ and $f''$ the real and imaginary parts of a complex function $f$, respectively, then we can write

$$\widetilde{G}_{c\hat{c}}''(\omega) = \alpha + \widetilde{G}_{c\hat{c}}''^{(\mathrm{reg})}(\omega), \tag{60}$$

where $\lim_{\omega\to\infty} \widetilde{G}_{c\hat{c}}''^{(\mathrm{reg})}(\omega) = 0$. From Eqs. (48), (49), and (52), we find that $\alpha^{-1} = a_0 + \lim_{\omega\to\infty} \widetilde{\Sigma}_{c\hat{c}}''(\omega)$. At the one-loop order,
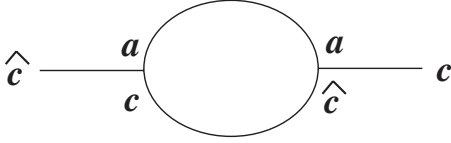
FIG. 3. The one-loop diagram contributing to $\widetilde{\Sigma}''_{c\hat{c}}(\omega)$ in the $\omega \rightarrow \infty$ limit

a nonvanishing contribution to $\widetilde{\Sigma}''_{c\hat{c}}(\omega)$ in the infinite-$\omega$ limit comes from the diagram depicted in Fig. 3. We thus have

$$\lim_{\omega \rightarrow \infty} \widetilde{\Sigma}''_{c\hat{c}}(\omega) = -\alpha \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \widetilde{G}_{aa}(\omega) = -\alpha G_{aa}(0), \quad (61)$$

and

$$\frac{G_{aa}(0)}{a_0^2} = \frac{1}{a_0 \alpha} - \frac{1}{(a_0 \alpha)^2} \quad (62)$$

is the initial value of the dimensionless correlation function $G_{aa}(t)/a_0^2$. Therefore, the initial value of the correlation function is determined self-consistently in the present field theoretic approach. This is in contrast to other approaches [14,17] where the static limit of the correlation functions was used as an input for the initial condition. Note that $G_{aa}(0)/a_0^2 \leq 1/4$ where the maximum value occurs when $a_0 \alpha = 2$. We believe that this peculiar behavior is due to the one-loop perturbation theory and that if we consider a higher-loop theory this condition will certainly change.

We now present numerical solutions to the above coupled equations. We start by evaluating the one-loop self-energies using Eqs. (53)–(59) from some appropriate initial form of the correlation functions (e.g., the bare correlation functions) given as functions of time. We make Fourier transforms of these self-energies and update the correlation functions by using Eqs. (44)–(48). Then, the inverse Fourier transforms are performed to compare the input and output correlation functions. This procedure is repeated until the convergence is achieved. We find that the convergence is achieved in less than 100 iterations in most cases. The number $N$ of mesh points used in the time and frequency integrals ranges from 8000 to 26 000. The cutoffs, $\Lambda_t$ and $\Lambda_\omega$, for the time and frequency integrals, respectively, must be adjusted so that all the five correlation functions and the seven self-energies are accommodated both in the time and frequency spaces. We maintain that $\Lambda_t \Lambda_\omega \sim N\pi$ to have a consistent numerical Fourier transform [29]. As we will explain below, $G_{aa}(t)$ can in general be a relatively long-ranged function, but other correlation functions such as $G_{c\hat{b}}(t)$ and $G_{c\hat{c}}^{(\text{reg})}(t)$ are short-ranged so that we need large $\Lambda_\omega$ for those functions. As functions of short and long-ranged are mixed in the calculations, both $\Lambda_t$ and $\Lambda_\omega$ must be sufficiently large.

For a numerical calculation of the above coupled set of equations, we need to put everything in dimensionless forms. From Eqs. (1) and (5), we see that $a_0/\Gamma$ has the dimension of time. Once we put all the correlation and response functions
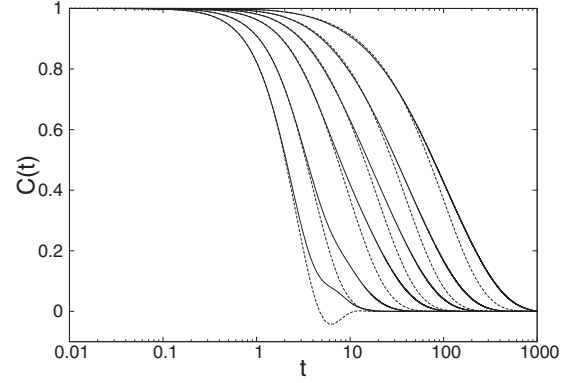


FIG. 4. Normalized correlation function $C(t)$ as a function of time $t$ measured in units of $a_0/\Gamma$ for fixed $\widetilde{T}/\kappa = 0.1$. The solid lines are the solutions to the self-consistent one-loop equations for $\widetilde{T} = 0.05$, 0.025, 0.01, 0.005, 0.0025, and 0.001 from left to right. The dashed lines are the corresponding bare correlation functions.

in their respective dimensionless forms, we find that the self-consistent equations are completely described by two dimensionless parameters, $\kappa$ and $\widetilde{T}$ defined by

$$\kappa \equiv \left( \frac{Ja_0}{\Gamma} \right)^2 a_0 A, \quad \widetilde{T} \equiv \left( \frac{J}{\Gamma} \right)^2 a_0 T. \quad (63)$$

In Figs. 4 and 5, we plot the normalized correlation function $C(t) \equiv G_{aa}(t)/G_{aa}(0)$ which corresponds to the density autocorrelation function in the FNH for various values of the parameters $\kappa$ and $\widetilde{T}$. These are compared with the corresponding bare correlation functions. From these figures, we can see that in general the one-loop correlation functions are more stretched in later times compared to the bare correlation functions. We perform numerical calculations for fixed value of $\widetilde{T}/\kappa = T/(a_0^2 A)$. The analytic expressions for the bare correlation functions can easily be obtained from the Gaussian part of Eq. (9), and we note that the initial value of the bare function $G_{aa}^{(0)}(t)$ is given by $G_{aa}^{(0)}(0)/a_0^2 = \widetilde{T}/\kappa$ in dimen-
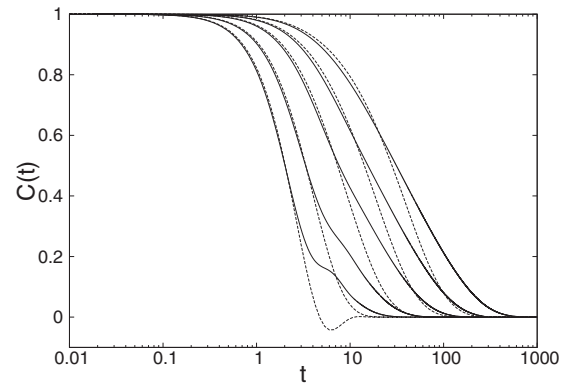


FIG. 5. Normalized correlation function $C(t)$ as a function of time $t$ measured in units of $a_0/\Gamma$ for fixed $\widetilde{T}/\kappa = 0.2$. The solid lines are the solutions to the self-consistent one-loop equations for $\widetilde{T} = 0.1$, 0.05, 0.02, 0.01, and 0.005 from left to right. The dashed lines are the corresponding bare correlation functions.

sionless quantities. As we can see by comparing Figs. 4 and 5, the difference between the renormalized correlation functions and the bare ones is small for small $\tilde{T}/\kappa$, but the nonlinear effects increase with increasing $\tilde{T}/\kappa$. The value of $a_0\alpha$ in Eq. (62) is also determined from the numerical calculation. For fixed $\tilde{T}/\kappa$, we find that $a_0\alpha$ and thus $G_{aa}(0)/a_0^2$ is almost constant when we change $\tilde{T}$. As we increase $\tilde{T}/\kappa$, the initial value $G_{aa}(0)/a_0^2$ increases as well. We find that the numerical solutions for the self-consistent equations exist only for $\tilde{T}/\kappa$ less than some maximum value which is found to be around 0.37 for the one-loop theory when $G_{aa}(0)/a_0^2$ reaches it maximum value 1/4. We note that this has nothing to do with the ENE singularity which we discussed earlier, since it involves the short-time behavior of the correlation functions. Indeed, as we approach this maximum value, the response functions, $G_{c\hat{b}}(t)$ and $G_{c\hat{c}}^{(\mathrm{reg})}(t)$ become increasingly short-ranged in time. For example, the initial time derivative $\dot{G}_{c\hat{b}}(0^+)$ approaches $-\infty$ as $\tilde{T}/\kappa$ approaches 0.37. We can understand these behaviors by investigating carefully the $t \rightarrow 0^+$ limit of the SD. Equation (24) [28]. We believe that the particular values of the parameters are specific to the one-loop calculation and will change as higher-loop contributions are considered.

As expected from the discussion in the previous section, $C(t)$ shows a completely ergodic behavior decaying to zero as $t \rightarrow \infty$ for all parameter values in our numerical calculations. We note that, in the standard MCT [1–4] above the ENE transition, the density autocorrelation function exhibits a plateau before its eventual decay in time. This is not obvious in our numerical results. We believe that the present toy model, without having realistic wavenumber dependence, is too simple to capture the plateau, if any, in the density autocorrelation function calculated from the one-loop approximation. In order to see if the DM field theoretical approach to FNH produces a plateau in a given order of the loop expansion, one would have to solve wavenumber-dependent versions of Eqs. (44)–(59). This remains to be seen in the future study.

## V. DISCUSSION AND CONCLUSION

In this paper, we have presented and studied the toy model of FNH of supercooled liquids containing only a couple of dynamical field variables $a(t)=a_0+\delta a(t)$ and $b(t)$ without any spatial dependence. We have developed two different field theoretic formulations of this model first by following the DM prescription and then the method involving a set linear FDR. We were also able to perform numerical calculations on the coupled equations for the correlation and response functions in the DM field theory at the one-loop order.

The major difference between the two field theoretic formulations is the way of treating the density nonlinearities that appears in the problem in the form of $1/a(t)$. In the DM field theory this is treated in a simple way by presenting a single auxiliary field $c(t)$ and its hatted counterpart $\hat{c}(t)$. On the other hand, in the field theory with linear FDR, a couple

of auxiliary fields $\eta(t)$ and $\theta(t)$, as well as their hatted partners, are presented in such a way that the dynamical action becomes invariant under a set of time-reversal transformations. This manipulation results in the dynamical action with terms which are nonpolynomial functions of the main field variables $\delta a(t)$ and $b(t)$. We note that these terms are proportional to $\hat{\eta}(t)$ and $\hat{\theta}(t)$ fields. Therefore, in the renormalized perturbation theory at a given order of the loop expansion, one has to truncate the nonpolynomial functions at the appropriate order. In some sense, we might say that, because of the truncation, the effect of the density nonlinearities would not be fully incorporated into the field theory at any finite order of the perturbation theory. We believe that this is one of the reasons why the sharp ENE-type transition appears at the one-loop order in this formulation.

This is in contrast to the DM field theory where no truncation of the dynamical action is necessary when the loop expansion is performed. The loop expansion, therefore, has a different meaning from the field theory with linear FDR. We have shown that the self-energy $\Sigma_{\hat{c}a}(t)$ which corresponds to the one that couples the density and the hatted velocity fields plays a key role in removing the sharp transition in the one-loop calculation. This is consistent with the original finding by DM [8] and with the recent nonperturbative analysis of the FNH [21]. This fact has often been interpreted as the coupling between the current and the density being responsible for the ergodicity restoring mechanism. In our toy model, the field $\hat{c}(t)$ is presented in the DM field theory to take care of the density nonlinearities as $\hat{\eta}(t)$ and $\hat{\theta}(t)$ are in the other formulation. Even though we only perform the one-loop calculation, we might say that the self-energy $\Sigma_{\hat{c}a}(t)$ contains the nonperturbative information arising from keeping all the terms in the expansion of the $1/a$ nonlinearities. It is, however, not obvious how the expansion of the $1/a$-type terms in the dynamic action is related to the field theoretic loop expansion in detail. To test more rigorously whether or not this kind of nonperturbative information is indeed captured in the order-by-order of field theoretic loop expansion of the DM type, it would be necessary to perform higher-loop order calculations and to study the existence of the ENE transition. This is left to the future work. In this respect, it may be more appropriate to regard the cutoff mechanism resulting from the full nonperturbative treatment of the density nonlinearities than from the coupling between the current and the density. The recent numerical calculation [22] where a direct integration of the generalized Langevin equations in FNH are performed also demonstrates that the $1/\rho$ nonlinearities ($\rho$ is the density) are playing an essential role restoring the ergodic behavior in supercooled liquids.

It is sometimes discussed in literatures that the DM field theory is inconsistent with the FDR. However, a set of FDR does hold in the DM field theory, which is given in Eq. (14) for our toy model. In the field theory with linear FDR, a larger set of FDR exists as in Eqs. (22) and (33). In the original analysis of 1986 [8], Das and Mazenko used another FDR (linking $G_{aa}$ and $G_{a\hat{a}}$ in the toy model notation) in addition to Eq. (14) and simplified the equations involved. This is valid only in the hydrodynamic limit. In the present paper, we do not use such additional simplifications. Instead,

we keep only Eq. (14), and take the equations which the correlation functions satisfy, Eqs. (44)–(59), as a set of self-consistent equations, and solve them numerically in Sec. IV. This program could be generalized to a realistic situation where the full spatial dependence is present. What we have shown in this paper is that the DM field theory, viewed as a collection of self-consistent equations for the correlation functions at a given order of loop expansion, is a well-defined field theoretic approach to glass forming liquids.

There are other field theoretic approaches to the FNH than those considered in this paper. A similar toy model of the FNH to ours but of different nature has been studied by Kim and Kawasaki [23] some time ago. In this model, the $N$ component of density-like and the $M$ component momentum-like field variables are presented without spatial dependence. They consider the limit where the numbers $N$ and $M$ approach infinity, and find that a sharp transition is present when the condition $M < N$ is maintained in the limiting process, while it is absent when $M = N$. In Ref. [9], a simpler version of the FNH than that of DM was considered, where the sharp transition was found to be absent. We note, however, that in both cases the effective free energy is quadratic in both density and the momentum variables from the outset. There is no need to introduce the auxiliary fields and a full set of linear FDR exist in these models. Therefore the kind of density nonlinearities discussed in this paper is not present and the absence of the sharp transition found in these works is probably of a different origin.

The work by Mayer *et al.* [30] is another interesting zero-dimensional model for glass forming liquids. It is in general hard to make a direct connection between the projection operator approach, in which Ref. [30] is set, and the field theoretical one. It is, however, clear from Ref. [30] that a non-perturbative effect that comes from considering an infinite number of equations is responsible for cutting off the sharp transition. In Ref. [30], the sharp transition is always present when one considers only a finite number of equations. This is related to our finding that, when the density nonlinearities are treated in the field theory with linear FDR within the loop expansion, one has to truncate a nonpolynomial function, and the sharp transition follows. We might say that the nonperturbative information that cuts off the sharp transition is somehow preserved in the DM approach, since one can avoid truncating the dynamical action at a given order of the loop expansion.

There are many ways in which the present result can be generalized. An obvious generalization is to consider a higher-order perturbation theory. Because of the simple nature of the model, one can without much difficulty construct the higher-loop DM field theory of the model and perform the numerical calculations as done in this paper. It will be interesting to see how the one-loop results, especially the particular initial values of the correlation functions, get changed when the higher-loop contributions are considered. We believe that the numerical methods developed in Sec. IV for treating the coupled equations for the correlation and response functions, especially those concerning the short-time behavior of functions, will prove to be useful for an eventual application to the full wavenumber dependent FNH in the future.

### ACKNOWLEDGMENTS

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